REPORT NO. 1.

December, 1946.

THE COLLEGE OF AERONAUTICS

CRANFIELD.

Ignoration of Distortional Co-ordinates in the Theory of Stability and Control

- By -

W. J. Duncan, D.Sc., M.I.Mech.E., F.R.Ae.S.,
Professor of Aerodynamics at the College
of Aeronautics, Cranfield.

-----------OoO------------

SUMMARY.

GATES AND LYON HAVE PROPOSED TO TREAT THEORETICALLY THE STABILITY AND CONTROL OF DEFORMABLE AIRCRAFT BY A METHOD IN WHICH THE DISTORTION CO-ORDINATES ARE IGNORED AND THE INFLUENCE OF DISTORTION IS ALLOWED FOR BY SUITABLE MODIFICATIONS OF THE DERIVATIVES AND OTHER COEFFICIENTS. IN THE PRESENT PAPER AN EXACT METHOD FOR ELIMINATING THE DISTORTION CO-ORDINATES IS GIVEN AND THE CONDITIONS IN WHICH THE TRUE ELIMINANT CONFORMS WITH THE SIMPLIFICATION OF GATES AND LYON ARE EXAMINED. IN GENERAL THE SIMPLIFICATION IS NOT JUSTIFIED MATHEMATICALLY, BUT IN CERTAIN CIRCUMSTANCES IT PROVIDES AN ACCEPTABLE APPROXIMATION. IT WILL NOT BE PRACTICALLY VALID UNLESS THE STRUCTURAL DISTORTIONS OCCUR SO RELATIVELY SLOWLY THAT THE ASSOCIATED INERTIA FORCES ARE NEGLIGIBLE, i.e. THE DISTORTIONS MUST BE QUASI-STATIC.

C.A.

Gates and Lyon and their collaborators [1, 2, 3], with the aim of Mathematical simplification, have proposed to treat theoretically the motions of a deformable aircraft by a method in which the deformation co-ordinates do not appear explicitly but are allowed for by modifying the coefficients of the usual dependent variables. The object of the present paper is to examine the validity of this method.

We shall suppose that, in the absence of distortion, the motion would be described by ℓ dependent variables γ_1, γ_2, etc., which will be called <u>body co-ordinates</u> ※. These are so chosen that their integrals do not appear in the dynamical equations, which are assumed to be second order differential equations with constant coefficients, some of which may be absent. The deformation <u>co-ordinates</u> are δ_1, δ_2, --- δ_m, so the total number of <u>co-ordinates</u> is

$$n = \ell + m. \qquad - \quad - \quad - \quad - \quad - \qquad (1)$$

When it is desired to refer to a co-ordinate without specifying its type it will be denoted as χ_r. Thus

$$\left.\begin{array}{l} \chi_r = \gamma_r \quad (r \leqq \ell) \\ \chi_r = \delta_{r-\ell} \,(r > \ell) \end{array}\right\} - \quad - \quad - \qquad (2)$$

The complete set of dynamical equations for free motions of the type considered are typified by that corresponding to χ_r which will be written

$$\left(a_{r1}\frac{d^2}{dt^2} + b_{r1}\frac{d}{dt} + c_{r1}\right)\chi_1$$
$$+ \left(a_{r2}\frac{d^2}{dt^2} + b_{r2}\frac{d}{dt} + c_{r2}\right)\chi_2$$
$$+ \cdots + \left(a_{rn}\frac{d}{dt^2} + b_{rn}\frac{d}{dt} + c_{rn}\right)\chi_n = 0, - \qquad (3)$$

or, more compactly, as

$$a_{r1}(D)\chi_1 + a_{r2}(D)\chi_2 + \cdots + a_{rn}(D)\chi_n = 0, - \qquad (4)$$

where $a_{rs}(D)$ is the differential operator

$$a_{rs}\frac{d^2}{dt^2} + b_{rs}\frac{d}{dt} + c_{rs}.$$

/In .....

※ These need not be Lagrangian co-ordinates and are often linear or angular velocities.

In the standard method for solving the dynamical equations it is assumed that χ_r is proportional to $\exp(\lambda t)$. The coefficients λ are the roots of the determinantal equation, and to each root λ corresponds a constituent motion. For conciseness write

$$a_{rs}(\lambda) \equiv a_{rs}\lambda^2 + b_{rs}\lambda + c_{rs} \quad - \quad - \quad - \quad (5)$$

Then, for the particular constituent motion considered, the typical dynamical equation (4) becomes

$$a_{r1}(\lambda)\chi_1 + a_{r2}(\lambda)\chi_2 + \cdots + a_{rn}(\lambda)\chi_n = 0 \quad (6)$$

where the symbols χ are new "amplitudes".

We shall show how the distortion co-ordinates can be eliminated and for the general case we shall use the notation of partitioned matrices on account of its clear conciseness. However, for the benefit of those who are unacquainted with matrices, we shall begin with the easy case where there is only one distortion co-ordinate δ_1 since here the matrix notation can be dispensed with. The dynamical equation corresponding to $\delta_1 \equiv \chi_n$ is the last of the set and for the constituent motion considered becomes (see equation (6))

$$a_{n1}(\lambda)\gamma_1 + a_{n2}(\lambda)\gamma_2 + \cdots + a_{n,n-1}(\lambda)\gamma_{n-1}$$
$$+ a_{nn}(\lambda)\delta_1 = 0 \quad - \quad - \quad - (7)$$

Hence $$\delta_1 = -\sum_{s=1}^{n-1} a_{ns}(\lambda)\gamma_s \Big/ a_{nn}(\lambda) \quad - \quad - \quad - (8)$$

This value may be substituted in the dynamical equations corresponding to the body co-ordinates γ, and these equations are then cleared of the distortion co-ordinate. The question is:

> In what circumstances are the resulting equations of the same degree in λ as when distortion is really absent?

In answering this question we shall assume that for each value of r up to $(n - 1)$ some at least of the expressions $a_{rs}(\lambda)$ are of the second degree in λ, i.e., are not degenerate.

In order to clear equation (6) of fractions after substituting from (8) for $\chi_n \equiv \delta_1$ it is necessary to multiply throughout by $a_{nn}(\lambda)$. This will result in raising the degree in λ of the equation unless $a_{nn}(\lambda)$ is independent of λ.

/ Even ....

Even when this condition is satisfied the distortion coupling term $a_{rn}(\lambda)\delta_1$ in the equation will, from (8), transform into

$$-a_{rn}(\lambda)\sum_{s=1}^{n-1} a_{ns}(\lambda)\gamma_s \Big/ c_{nn} \qquad - \quad - \quad - (9)$$

The coefficient of γ_s in the last is the product of two terms each of which represents coupling between the distortion and body co-ordinates. The product will be of a degree in λ higher than the second unless both $a_{rn}(\lambda)$ and $a_{ns}(\lambda)$ are restricted to be <u>linear</u> in λ, or unless one degenerates to a mere constant. Hence the strict conditions for the elimination of the distortion co-ordinate without raising the degree of the equations are:

1. $a_{nn}(\lambda)$ must be a mere constant

2. the quadratic or inertia coefficients must be absent from all the terms representing couplings between the general and distortion co-ordinates. Alternatively, one of the pair of coupling terms $a_{rn}(\lambda)$ and $a_{ns}(\lambda)$ may be quadratic when the other is a mere constant.

Now condition 1. can never be satisfied strictly because $a_{nn}(\lambda)$ contains a direct inertia term which is necessarily finite and positive ᴷ. However, when the modulus of λ for the constituent motion considered is much less than the moduli of the roots of

$$a_{nn}\lambda^2 + b_{nn}\lambda + c_{nn} = 0 \qquad - \quad - \quad - (10)$$

it will be a legitimate approximation to replace $a_{nn}(\lambda)$ by the constant c_{nn}. Also, it may be legitimate to disregard such inertial couplings between distortion and body co-ordinates as exist when the modulus of λ is relatively small.

For the general case where there are m distortion co-ordinates we shall have, in matrix notation,

$$a(\lambda)\chi = 0 \qquad - \quad - \quad - \quad - (11)$$

which may be written in the partitioned form

$$\begin{bmatrix} b(\lambda) & c_1(\lambda) \\ c_2(\lambda) & d(\lambda) \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = 0 \qquad - \quad - \quad - (12)$$

/Here .....

ᴷ In the absence of massless structural material !

Here $b(\lambda)$ and $d(\lambda)$ are square matrices of order ℓ and m respectively, and correspond to body motions without distortion and pure distortions respectively. The couplings of the two kinds of motion are represented by the matrices $c_1(\lambda)$ and $c_2(\lambda)$, which are of the orders $\overline{\ell m}$ and $\overline{m \ell}$ respectively. Equation (12) can be expanded to

$$b(\lambda)\gamma + c_1(\lambda)\delta = 0 \quad - \quad - \quad - \quad (13)$$

$$c_2(\lambda)\gamma + d(\lambda)\delta = 0 \quad - \quad - \quad - \quad (14)$$

The last gives

$$\delta = -d^{-1}(\lambda)c_2(\lambda) \quad - \quad - \quad - \quad (15)$$

and on substitution (13) becomes

$$\left\{b(\lambda) - c_1(\lambda)d^{-1}(\lambda)c_2(\lambda)\right\}\gamma = 0 \quad - \quad (16)$$

Equation (16) contains the body co-ordinates only and may be written concisely as

$$\beta(\lambda)\gamma = 0 \quad - \quad - \quad - \quad - \quad (17)$$

Hence the influence of distortion is exactly taken into account on replacing $b(\lambda)$ by

$$\beta(\lambda) \equiv b(\lambda) - c_1(\lambda)d^{-1}(\lambda)c_2(\lambda) \quad - \quad (18)$$

In order that the procedure of Gates and Lyon shall be legitimate it is necessary that, for all the values of λ considered, $\beta(\lambda)$ shall not be of higher degree in λ than $b(\lambda)$, when cleared of fractions. This condition will not, in general, be satisfied unless the determinant

$$|d(\lambda)|$$

reduces to a mere constant. Such a reduction cannot, in general, occur, but it may be a legitimate approximation to treat $d^{-1}(\lambda)$ as a matrix of constants when the modulus of λ is much smaller than the moduli of all the roots of

$$|d(\lambda)| = 0 \quad - \quad - \quad - \quad - \quad - \quad (19)$$

In order that

$$c_1(\lambda)d^{-1}(\lambda)c_2(\lambda)$$

shall not be of degree in λ higher than the second, it will be necessary, in general, that either

(a) $c_1(\lambda)$, and $c_2(\lambda)$ shall not be of degree exceeding unity, i.e. the inertia terms must be negligible for the values of λ considered.

/or (b) .....

or (b) one of these coupling matrices is effectively independent of λ

CONCLUSION.

The practical validity of the procedure of Gates and Lyon must be examined in each case, but one necessary condition for its applicability is that the values of λ for the motions considered should be small in relation to those which characterise free motions of pure distortion. In other words, the loads causing distortion must be applied so relatively slowly that they are quasi-static.

REFERENCES

1. GATES, S.B. and LYON, H.M. — " A Continuation of Longitudinal Stability and Control Analysis. Part I. General Theory." R.A.E. Report Aero.1912, Feb.1944 A.R.C. 7687.

2. GATES, S.B. and LYON, H.M. — Ditto, Part II. " Interpretation of Flight Tests." R.A.E.Report Aero.1953, Aug.1944 A.R.C. 8108.

3. LYON, H.M. and RIPLEY, J. — " A General Survey of the Effects of Flexibility of the Fuselage, Tail Unit and Control Systems on Longitudinal Stability and Control." R.A.E. Report Aero.2065, July 1945 A.R.C. 9015.